International Journal of Social Science and Economic Research

ISSN: 2455-8834

Volume:06, Issue:10 "October 2021"

HOW TO USE THE MARKOWITZ MEAN-VARIANCE OPTIMIZATION TO DIVERSIFY THE RISK OF A PORTFOLIO WHILE MAXIMIZING THE EXPECTED RETURNS

Jiayang Wen[1]

Supervising Teacher: Niyu Jia[2]

[1]Beijing Huijia Private School

[2]Boston University, U.S.

DOI: 10.46609/IJSSER.2021.v06i10.020 URL: https://doi.org/10.46609/IJSSER.2021.v06i10.020

ABSTRACT

Mean-Variance Model (Modern portfolio theory) maybe the most famous model in financial field. It assesses a portfolio which's the expected return (mean) is maximized under a given risk (variance). It comes from assumption that investor want as high as return while as low as risk as he could when he invested a couple of assets (a portfolio is the collection of many assets). This model could give us many optimal portfolio (efficient portfolio frontier) when every asset's expect return and their covariance matrix are known. The accuracy estimating the covariance matrix is the most essential part implementing portfolio optimization.

Thus, in this project, we will perform the mean variance portfolio of the targeted portfolio with Ledoit-Wolf shrinkage methodology which can give us robust estimation of covariance matrix. Then we will use the optimal portfolio to visualize the efficient frontier and compare the optimal portfolio with index or other randomly chosen portfolio.

Keywords: CAPM, Risk minimization, Ledoit-Wolf shrinkage, Chinese stocks

1. Introduction

In the recent decades, there is an increasingly trend of applying capital asset pricing model (CAPM) by the individual investor to assist their portfolio construction. This project provides an introduction to mean-variance analysis and CAPM. We begin with the mean-variance analysis of Markowitz (1952) when there is no risk-free asset also discuss the difficulties of implementing mean-variance analysis in practice and outline some approaches for resolving these difficulties. Because optimal asset allocations are typically very sensitive to estimates of expected returns and covariance, these approaches typically involve superior or more robust parameter estimation

ISSN: 2455-8834

Volume:06, Issue:10 "October 2021"

methods. Mean-variance analysis leads directly to the capital asset pricing model or CAPM. The CAPM is a one-period equilibrium model that provides many important insights to the problem of asset pricing. The language / jargon associated with the CAPM has become ubiquitous in finance.

(1). Financial Concepts

- Expected Return: The expected return on portfolio assets is a weighted average of the returns of each asset in the portfolio, often used with a log returns and simple rates of return. Since the return of financial assets are mostly in the distribution pattern of peaks and thick tails, the quantitative models mostly use the log return.

- Volatility: Volatility is used to measure the risk of an asset, the standard deviation of the return sequence. For portfolios, volatility is the standard deviation of the return on the whole portfolio.

- Assets Correlation: If there is a positive correlation between the two assets (correlation coefficient $\rho>0$), the price and rate of return between the two assets will change in the same direction, and if there is a negative correlation (correlation factor $\rho<0$), the price and rate of return between the two assets will change in reverse." As a result, negative-related assets tend to hedge some of therisk, reducing the overall risk to the portfolio.

- Sharpe Ratio: That is, the profitability of the asset, calculated as:

  I. Sharpe = (expected return - risk-free interest rate) / volatility

  II. The higher the Sharp ratio, the greater the benefit of a unit of risk.

(2). About Investors

- Utility Function: The utility function is a function on $R \rightarrow R$, and if the wealth isw, U(w) represents the utility (satisfaction) that the investor obtains from w; Typically, the utility function U (w) should meet the following criteria:

  I. The more wealth you have, the more utility you have, i.e. $U'(w) >0$

  II. Increased wealth and decreasing marginal utility, i.e. $U''(w) <0$

(eg: Wealth increases from $1 to $2 and the utility is greater than wealth increases from $100 to $101)

Generally, the utility function will look like as this type of shape:

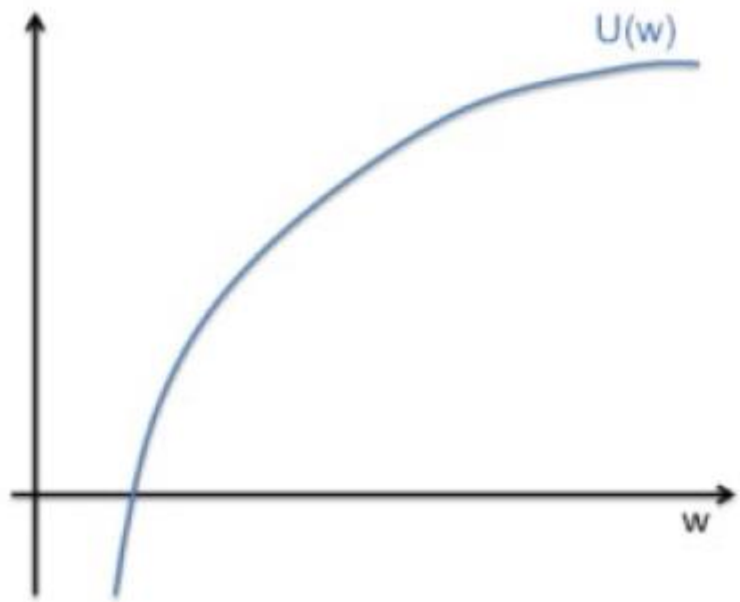

Fig.1 Utility function normal shape

Risk Aversion: Starting from utility function, we can define an investor's risk aversioncoefficient. Risk aversion coefficient is an extremely important parameter in asset allocation, which reflects the investor's personalized tolerance to risk. The use of 'a' to represent the risk aversion factor represents an increase in the minimum expected rateof return required by investors for each more unit of risk assumed.

eg: Let risk and expected return be used as the minimum unit of 1%, then a=5 means that investors require a return increase of at least 5% to be willing to take anadditional 1% risk.

The greater the risk aversion factor, the more conservative the investor is. In general, the risk aversion factor should be greater than zero.

2. Portfolio Construction

This portfolio includes 6 stocks with great performance and potential from different promising industries. These 6 stocks can be divided into the following four major categories: the vertical industry of electric car, Medicare industry, high-tech industry and the transportation industry. Since electric car is one of the most potential industries in the long-term, a great portion of our portfolio's money is allocated to the two leading companies in each sector of the whole industry

ISSN: 2455-8834

Volume:06, Issue:10 "October 2021"

chain. The bidding on the whole industry further enables us to diversify our risk and seize the opportunities of long-term returns. Moreover, sometimes the stocks' price will not reflect the true value but fluctuate in the short-term. Thus, in order to minimize both the long-term and short-term risk caused by the companies' development and the sentiment of the market, we decide to allocate into four different promising industries that will help us to diversify the short-term fluctuation due to the market sentiment to a specific industry. Furthermore, we set a boundary weight for each stock in the portfolio based on the risk aversion of our clients to make appropriate adjustment.

Below are the 6 stocks in the portfolio

1. Contemporary Amperex Technology Co., Limited (300750)
2. BYD Company Limited (002594.SZ)
3. Aier Eye Hospital Group Co., Ltd. (300015)
4. S.F. Holding Co., Ltd. (002352)
5. Cambricon Technologies Corporation Limited (688256.SS)
6. BOE Technology Group Company Limited (000725.SZ)

Contemporary Amperex Technology Co., Limited (300750)

Contemporary Amperex Technology Co., Limited (CATL) is a global leader in the development and manufacturing of lithium-ion batteries, with businesses covering R&D, manufacturing and sales in battery systems for new energy vehicles and energy storage systems. The company has so far ranked as No.1 globally in EV battery consumption volume for four consecutive years which demonstrates its unparalleled competitive edge in the industry. Since 2011, CATL has already formed a complete closed-loop industrial chain of materials, batteries, modules, systems, and recycling.

In the first three quarters of 2020, although the performance of CATL has fluctuated slightly due to the impact of the domestic epidemic and the industry decline, it still achieved operating income of 31.522-billion-yuan, net profit of 3.357 billion yuan, and achieved 18.25GWh power battery installed capacity with a market share of 47.57% ranking first in the industry, which all show that the pandemic has little impact on its business.

ISSN: 2455-8834

Volume:06, Issue:10 "October 2021"

In the long-term, CATL leads the industry in R&D investment and the number of patents. From the perspective of the number of patents and technical team, as of the first half of 2020, CATL has 2,642 domestic patents, 196 overseas patents, and 5,368 R&D technicians, providing patents and personnel protection for upstream and downstream cooperative R&D and technical reserves. Among peer companies, CATL is one of the few companies whose R&D expenditures are fully expensed in the current period.

Since not only does CATL have a great resistance to the short-term fluctuation, but also it has a promising and sustainable growth in the long-term given its investment inR&D, CATL becomes one of the few best stocks selected in our portfolio.

Lepu Medical Technology (Beijing) Co., Ltd. (300003)

Lepu Medical, a leading cardiovascular enterprise, is one of the few domestic high- end medical equipment companies that can form strong competition with imported products. Its main business includes medical equipment, medical services, medicines and new-type medical treatments which has an enormous potential market in the future. Not only it has the largest market share in the heart health industry, but also it has been keeping a high speed of growth. In the seven years from 2013 to 2019, the company's total revenue has an average compound annual growth rate of 29.02%.

In the future, there will be an increasing trend of population aging in China, causing cardiovascular diseases a prevailing diseases people need to take care of, which will render the Lepu medical a great market and opportunity. According to the "China Cardiovascular Report 2018", there are 290 million patients with cardiovascular diseases in China, of which hypertension accounts for 245 million. China's annual deaths from cardiovascular diseases account for 44.33% of the total deaths of urban and rural residents, which is the first among all diseases. Thus, the focuses on medical equipment and medical services and the dominating market share of Lepu medical will form a natural moat for this company in the long-term development.

S.F. Holding Co., Ltd. (002352)

Speedy Express is the dominating company in the express industry in China for years. There are several reasons that explain the strong business moat of the SF. First of all, SF Land Network is composed of tens of thousands of business outlets across the country, automated transit yards, extensive transportation network, huge storage area, and humanized customer service that other companies could not compare with. The scale of SF's infrastructure construction and fixed assets is far ahead of other express companies. Secondly, SF Express is the domestic cargo airline with

ISSN: 2455-8834

Volume:06, Issue:10 "October 2021"

the largest number of all-cargo aircrafts. SF Airlines currently operates a fleet of 73 all-cargo aircrafts. Among them, SF Airlines has expanded to 59 all-cargo aircrafts, far surpassing other cargo airlines which firmly consolidate its dominating strategy. Thirdly, SF Express independently developed a complete smart logistics platform to support business development, while applying data mining, machine learning, statistical analysis and other technological methods to actual business scenarios to enhance the company's technological competitiveness. For example, in terms of smart warehouse network, the company has built a complete SF Cloud warehouse information system, in terms of terminal collection and dispatch intelligence; it continues to optimize convenient interaction with customers on the client side, etc.

When people compare the SF express with other giant express company such as UPS, it’s not hard to recognize that there is still a great gap between, which in other word, there is a still prominent growth of SF in the express market. For example, in 2019, the revenue of SF express is only 112.2 billion RMB while the revenue of UPS is 518.7 billion RMB. According to some experts’ prediction, China might replace America to become the largest economy entity during 2030 ~ 2035, which means that the express market size in China will further grow in the future that renders a greater potential growth for SF.

Cambricon Technologies Corporation Limited (688256.SS)

Cambricon focuses on a new intelligent ecosystem integrating cloud and edge, and is committed to build core processor chips for various types of intelligent cloud servers, intelligent edge devices, and intelligent terminals, so that machines can better understand and serve humans. As a chip design company that aimed at AI earlier, Cambricon has gathered a variety of high-quality resources and attaches great importance to technology. Technology research and development and productization are implemented, and it has hard-core architecture technology. The company develops for different demand scenarios such as the terminal, edge, and cloud series chips which have all been released as the first one among the world that could be produced in a large-scale. From the perspective of demand side, digitalization changes from the quantity explosion to qualitative explosion, so the demand for AI processing will greatly increase. In the digital age with explosive growth of information, AI can greatly increase the value of data. The rapid popularity of various AI applications has driven the demand for AI chips to increase. Second, as the portal for human-computer interaction increases, the penetration rate of AI chips in terminal equipment is expected to increase from 16% to 80%. Based on the data from Tractica, the AI chip market will grow to 493 billion RMB in 2025 which provide the fundament for a sustainable growth for Cambricon. In addition, since many domestic AI-related companies are marked in the US "entity list", domestic AI chips are in urgent need which provide a great

ISSN: 2455-8834

Volume:06, Issue:10 "October 2021"

opportunity for Cambricon to replace the use of foreign products and increase its market share in domestic market.

Aier Eye Hospital Group Co., Ltd. (300015)

Aier Eye Hospital Group Co., Ltd. is an ophthalmology medical group with leading hospital scale and medical capabilities in China and around the globe. The company is mainly engaged in the diagnosis and treatment of various ophthalmic diseases, surgical services, and medical optometry. With its unique "tiered chain" development model and its supporting management system, the medical network has spread throughout China, Hong Kong, Europe, the United States, and Southeast Asia, and have established the strategic pattern of global development. At present, the company has more than 600 eye hospitals and centers, including more than 500 in Mainland China, 7 in Hong Kong, 1 in the United States, more than 80 in Europe, and 12 in Southeast Asia. In addition, the company continues to improve the construction of the medical research platform, and further promotes the coordinated development with other famous universities and institutions. Due to the rapid increase in the myopia rate of young people in China and the aging problem, the ophthalmology industry itselfhas a growth rate of more than 10% annually.

The future potential developments of Aier mainly come from the following aspects:

1. The national strategy for the prevention and control of young people's myopia has been improved which keeps the market at a large scale. (According to the 2019 annual report, the optics business is the fastest growing segment.)

2. The outbreak of cataracts and senile fundus diseases caused by aging problem will further expand the sales of Aier eye's products and services.

3. The increase in medical payment capacity brought about by the improvement of residents' living standards, and the increase in medical consumption from the upgrade will keep the market in a large size in the future that create a giant room for Aier eye to grow.

4. Based on the continuous investment on R&D and its huge market share (which causes the effect of economic of scale), Aier will has an increasing net profit due to an increase in operating efficiency, and a decrease in unit cost.

BOE Technology Group Company Limited (000725.SZ)

ISSN: 2455-8834

Volume:06, Issue:10 "October 2021"

BOE is the undoubted leadership in the LCD industry due to the decrease in supply side and increase in demand side. On the supply side, the large-size LCD panel industry has experienced continuous price changes for several years, which ushered in the withdrawal of some production capacity. Foreign companies such as Samsung SDC and LGD continue to reduce the LCD production line, which optimizes the supply side of the entire industry, especially for the mainland manufacturers.

According to Aijiwei's "2020-2021 Annual Special Report", the display industry was reshuffled globally in 2020, 16 LCD panel factories were sold or closed. In 2019, the global market share of mainland Chinese manufacturers will reach to 42.3%, becoming the world's largest panel manufacturing base. At that time, BOE's market share will reach 50% in the whole industry. Meanwhile, due to the industry development over the past decades that enables the localization of raw materials and machines, the LED manufacturers can further optimize their costs. On the demand side, the need of digitalization in the post-epidemic era is an irreversible trend around the globe. In accordance with the "Deloitte Technology Trends in 2021" based on the observation of the past 1 year's chain reaction of pandemic on businesses, the global enterprises are accelerating their digitalization to build "resilience" and create a comprehensive new business model. Online education and online entertainment triggered by the epidemic will generate explosive growth of application scenarios such as music, telecommuting, and telemedicine, new technologies such as 5G and AI.

The integration of technology and traditional industries is accelerating, and the market is in a strategic opportunity brought by digital transformation. Therefore, the profitability of domestic panel manufacturers will continue to increase. For instance, in the first quarter of 2021, BOE achieved revenue of 49.655 billion yuan, a year-on- year increase of 107.87%, and realized a net profit of 5.182 billion yuan, a year-on- year increase of 814.46%.

3. Methodology

Mean-Variance Optimization Model

The core of mean variance optimization is the convex optimization. In order to show the details of the methodology, we will first start with portfolio only contains 2 risky assets and then use the matrix format to denote the n risky asset cases.

ISSN: 2455-8834

Volume:06, Issue:10 "October 2021"

Two Risky Assets

Notation	Meaning
R_a	$Return\ of\ asset\ a$
R_b	$Return\ of\ asset\ b$
$\bar{R}_a$	$Expected\ (mean)\ return\ on\ asset\ a$
$\bar{R}_b$	$Expected\ (mean)\ return\ on\ asset\ b$
$\bar{R}_p$	$\text{Expected (mean) return on asset } p$
ω_a	$Weight\ in\ the\ portfolio\ of\ asset\ a$
ω_b	$Weight\ in\ the\ portfolio\ of\ asset\ b$
σ_a^2	$The\ varaince\ of\ the\ return\ on\ asset\ 1$
σ_b^2	$The\ varaince\ of\ the\ return\ on\ asset\ 2$
σ_{ab}	$The\ varaince\ of\ the\ return\ on\ assets\ a\ and\ b$
σ_p^2	$The\ varaince\ of\ the\ return\ on\ the\ porfolio$

Since all the wealth is invested into the two assets, $wa + wb = 1$, the expected return of the portfolio is a linear combination of the expected return of each asset:

$$\overline{R_a} = w_a \overline{R_a} + (1 - w_a)\overline{R_b}$$

The variance of the return on this portfolio is given by:

$$\sigma_p^2 = E\left[R_p - \bar{R}_p\right]^2$$

By expanding this expression, we get:

$$\sigma_p^2 = E[\omega_a R_a + (1 - w_a)R_b - (\omega_a \bar{R}_a + |(1 - w_a)\bar{R}_b)]^2 \qquad \text{(A.1)}$$

$$= E[\omega_a(R_a - \bar{R}_a) + (1 - w_a)(R_b - \bar{R}_b)]^2$$

$$= E[\omega_a^2(R_a - \bar{R}_a)^2 + (1 - w_a)^2(R_b - \bar{R}_b)^2 + 2\omega_a(R_a - \bar{R}_a)(R_b - \bar{R}_b)]^2$$

Given that $\mathrm{E}(R_a - \bar{R}_a)^2$ and $\mathrm{E}(R_b - \bar{R}_b)^2$ are the *variances* of the returns on assets a and b, respectively, while $\mathrm{E}[(R_a - \bar{R}_a)(R_b - \bar{R}_b)]$ is their *covariance*.

ISSN: 2455-8834

Volume:06, Issue:10 "October 2021"

We can therefore write:

$$\sigma_p^2 = [\omega_a^2\sigma_a^2 + (1 - w_a)^2\sigma_b^2 + 2\omega_a(1 - w_a)\sigma_{ab}] \qquad (A.2)$$

And

$$\sigma_p = [\omega_a^2\sigma_a^2 + (1 - w_a)^2\sigma_b^2 + 2\omega_a(1 - w_a)\sigma_{ab}]^{\frac{1}{2}} \qquad (A.3)$$

The covariance can be expressed as the product of the standard deviation of eachasset and the correlation coefficient between the assets (ρ_{ab}) i.e. $\sigma_{ab} = \sigma_a\sigma_b\rho_{ab}$.

We can therefore write (A.2) as

$$\sigma_p^2 = [\omega_a^2\sigma_a^2 + (1 - w_a)^2\sigma_b^2 + 2\omega_a(1 - w_a)\sigma_a\sigma_b\rho_{ab}] \qquad (A.4)$$

$$\sigma_p = [\omega_a^2\sigma_a^2 + (1 - w_a)^2\sigma_b^2 + 2\omega_a(1 - w_a)\sigma_a\sigma_b\rho_{ab}]^{\frac{1}{2}} \qquad (A.5)$$

It's easy to recognize that the portfolio risk depends both on the risk of the individual assets and on the correlation of their returns. This is what gives rise to a *diversification effect*. To examine this more closely, we begin with the case of *perfect positive correlation*.

Let's substitute ρ_{ab} = 1 into (A.4) to get:

$$\sigma_p^2 = [\omega_a^2\sigma_a^2 + (1 - w_a)^2\sigma^2{}_b + 2\omega_a(1 - w_a)\sigma_a\sigma_b] \qquad (A.6)$$

which corresponds to

$$\sigma_p^2 = [w_a\sigma_a + (1 - w_a)\sigma_b]^2$$

Hence

$$\sigma_p = w_a\sigma_a + (1 - w_a)\sigma_b \qquad (A.7)$$

The standard deviation of the portfolio is now a simple weighted average of the standard deviation of each asset, and there is no *diversification effect*. However, this only applies in the extreme case of a perfect positive correlation does. We see this from the fact that any $\rho_{ab} < 1$ would make the right-hand side of Eq. (A.4) smaller than (A.6).

ISSN: 2455-8834

Volume:06, Issue:10 "October 2021"

Now consider a case that the correlation between the two assets equals to $0(\rho_{ab} = 0)$.

Equation (A.4) then be simplified to

$$\sigma_p^2 = \omega_a^2\sigma_a^2 + (1 - w_a)^2\sigma_b^2 \qquad (A.8)$$

And

$$\sigma_p = [\omega_a^2\sigma_a^2 + (1 - w_a)^2\sigma_b^2]^{\frac{1}{2}} \qquad (A.9)$$

σ_p is now smaller than $w_a\sigma_a + (1 - w_a)\sigma_b$ unless $w_a = 1$ *or* 0, in which case the portfolio would be concentrated in either asset a or b, which would obviously exclude any *diversification effect*.

Finally, let's consider the case of a perfect negative correlation ($\rho_{ab} = -1$) into Eq. (A.4)

we get

$$\sigma_p^2 = \omega_a^2\sigma_a^2 + (1 - w_a)^2\sigma_b^2 - 2\omega_a(1 - w_a)\sigma_a\sigma_b$$

which corresponds to

$$\sigma_p^2 = [w_a\sigma_a - (1 - w_a)\sigma_b]^2 \qquad (A.10)$$

Or to

$$\sigma_p^2 = [-w_a\sigma_a + (1 - w_a)\sigma_b]^2 \qquad (A.11)$$

Hence

$$\sigma_p = w_a\sigma_a - (1 - w_a)\sigma_b \qquad (A.12)$$

Or

$$\sigma_p = -w_a\sigma_a + (1 - w_a)\sigma_b \qquad (A.13)$$

Either (A.12) or (A.13) is valid depending on which of the two is positive. In thiscase, it is (in principle) possible to entirely eliminate the risk of the portfolio. To find

ISSN: 2455-8834

Volume:06, Issue:10 "October 2021"

the w_a (and, by implication, w_b) necessary for this, set (A.13) equal to zero and solve for w_a to get

$$w_a = \frac{\sigma_a}{\sigma_a + \sigma_b} \quad \text{(A.14)}$$

The effect of diversification in the two assets case is illustrated in Fig. A.1, instylized form.

We saw that with a perfect positive correlation, both the risk and the return of the portfolio become linear combination of the two assets in the portfolio.

Appendix: The Mechanics of Mean-Variance Optimization

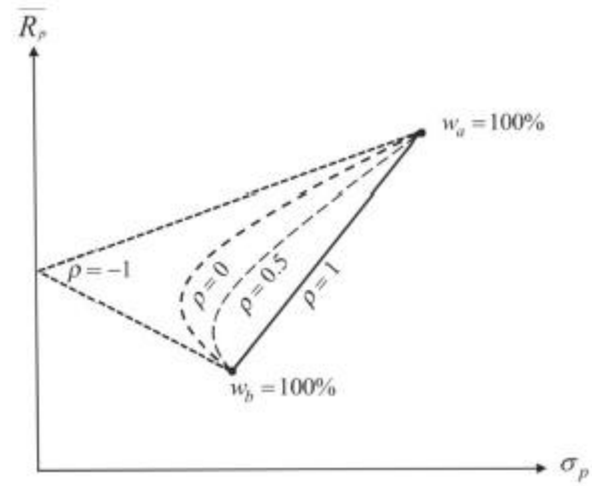

Fig.2 Correlation and diversification

For correlations lower than 1, however, the diversification effect is represented bythe non-linear combinations of assets a and b in ***Fig.2*** and, in the extreme case of $\rho_{ab} = -1$, by the straight lines to the vertical axis. Comparing the linear combination of $\rho_{ab} = 1$ with all the others, we see that diversification reduces the level of risk of the portfolio for a given level of return or, which is the equivalent, increase the level of return for a given level of risk.

To find the minimum variance portfolio in the two-asset case, we differentiate (A.4), which is $\sigma_p^2 = \omega_a^2\sigma_a^2 + (1 - w_a)^2\sigma_b^2 + 2\omega_a(1 - w_a)\sigma_a\sigma_b\rho_{ab}$,

with respect to w_a and set it equal to zero:

ISSN: 2455-8834

Volume:06, Issue:10 "October 2021"

$$\frac{\partial \sigma_p^2}{\partial w_a} = 2w_a\sigma_a^2 - 2\sigma_b^2 + 2w_a\sigma_b^2 + 2\sigma_a\sigma_b\rho_{ab} - 4w_a\sigma_a\sigma_b\rho_{ab}$$

Solving for w_a, we get:

$$w_a = \frac{\sigma_b^2 - \sigma_a\sigma_b\rho_{ab}}{\sigma_a^2 + \sigma_b^2 - \sigma_a\sigma_b\rho_{ab}} \qquad \text{(A.15)}$$

This is the general formula which, as we saw, becomes $\frac{\sigma_a}{\sigma_a + \sigma_b}$ when $\rho_{ab} = -1$.

We see directly that for uncorrelated assets ($\rho_{ab} = 0$), (A.15) is reduced to

$$w_a = \frac{\sigma_b^2}{\sigma_a^2 + \sigma_b^2} \qquad \text{(A.16)}$$

Portfolios of n Risky Assets

Generalizing to n assets, we can express the expected return on the portfolio as:

$$\bar{R}_p = \sum_{i=1}^{n} w_i \bar{R}_i \qquad \text{(A.17)}$$

Which is a simple weighed average of the return on each asset with $\sum_{i=1}^{n} w_i \bar{R}_i = 1$.

Portfolio variance equals

$$\sigma_p^2 = [\omega_a^2\sigma_a^2 + (1 - w_a)^2\sigma_b^2 + 2\omega_a(1 - w_a)\sigma_{ab}]$$

$$\sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_i \sigma_{ij} \qquad \text{(A.18)}$$

The double summation implies that we are adding up n^2 numbers. Each number corresponds to a pair of values for I and j in the equation. The variance of the portfolio corresponds to the weighted average covariance of these pairs. Writing thisin matrix form we get:

ISSN: 2455-8834

Volume:06, Issue:10 "October 2021"

$$\sigma_p^2 = \begin{bmatrix} w_1 w_1 \sigma_{11} & w_1 w_2 \sigma_{12} & \cdots & w_1 w_n \sigma_{1n} \\ w_2 w_1 \sigma_{21} & w_2 w_2 \sigma_{22} & & w_2 w_n \sigma_{2n} \\ \vdots & & \ddots & \vdots \\ w_n w_1 \sigma_{n1} & w_n w_2 \sigma_{n2} & \cdots & w_n w_n \sigma_{nn} \end{bmatrix} \quad \text{(A.19)}$$

Because the covariance of each asset with itself is equal to its variance, that is, $\sigma_{iiii} = \sigma^2$, we can decompose (A.18) into

$$\sigma_p^2 = \sum_{i=1}^{n} w_i{}^2 \sigma_i^2 + \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} w_i\, w_i \sigma_{ij} \quad \text{(A.20)}$$

The first term corresponds to the sum of the n *variances*, while the second term corresponds to the sum of the $n^2 - n$ covariances. Accordingly, our matrix will now read:

$$\sigma_p^2 = \begin{bmatrix} w_1{}^2 \sigma_2^2 & w_1 w_2 \sigma_{12} & \cdots & w_1 w_n \sigma_{1n} \\ w_2 w_1 \sigma_{21} & w_2{}^2 \sigma_2^2 & & w_2 w_n \sigma_{2n} \\ \vdots & & \ddots & \vdots \\ w_n w_1 \sigma_{n1} & w_n w_2 \sigma_{n2} & \cdots & w_n{}^2 \sigma_n^2 \end{bmatrix} \quad \text{(A.21)}$$

The diagonal from the upper left to the bottom right represents the weighted caraînces, while the off-diagonals represent the weighted covariances. Note that since $w_{ii} w_{jj} \sigma_{iijj} = w_{jj} w_{ii} \sigma_{jjii}$ the matrix is symmetric and each weighted covariance appears twice. The number of computations is therefore limited to n variances $+ \frac{n^2 - n}{2}$

covariances. Maintaining the matrix notation, we can rewrite (A.21) as the followingproduct:

$$\sigma_p^2 = \omega' \sum \omega \quad \text{(A.22)}$$

Where ω' = the transpose of the weight vector, $\sum$ = the variance-covariance matrix and ω = the weight vector. In expanded form this give us:

ISSN: 2455-8834

Volume:06, Issue:10 "October 2021"

$$\sigma_p^2 = [\omega_1 \omega_2 \cdots \omega_n] \begin{bmatrix} \sigma_2^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & & \sigma_{2n} \\ \vdots & & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{bmatrix}$$

The Limits to Diversification

Our previous analysis suggests that including more assets in the portfolio would increase the scope for diversification, if the assets involved are note perfectly positively correlated. This is generally tree, but note that the diversification effect will eventually be exhausted. To illustrate, suppose that the n assets are equally weighted in the portfolio, so that $w_i = \frac{1}{n}$ We can now rewrite (A.20) as

Appendix: The Mechanics of Mean-Variance Optimization

$$\sigma_p^2 = \sum_{i=1}^{n} \left(\frac{1}{n}\right)^2 + \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{i=1}^{n} \left(\frac{1}{n}\right)\left(\frac{1}{n}\right) \sigma_{ij} \qquad \text{(A.23)}$$

Factoring out $\frac{1}{n}$ from the first term of the equation and $\frac{n-1}{n}$ from the second, we

Get:

$$\sigma_p^2 = \frac{1}{n} \sum_{i=1}^{n} \left[\frac{\sigma^2}{n}\right]^2 + \frac{n-1}{n} \sum_{i=1}^{n} \sum_{i=1}^{n} \left[\frac{1}{n(n-1)} \sigma_{ij}\right] \qquad \text{(A.24)}$$

The first and second term in the square brackets represent the average of the n variances and the average of the $\frac{n-1}{n}$ covariances, respectively. As $n \to \infty$, the first term in (A.24) approaches zero, implying that all asset specific, or entirely independent assets ($\sigma_{iijj} = 0$), the covariance of the overall portfolio would eventually be reduced to zero. In the more realistic case of a positive covariance, the variance of the portfolio approaches the *average covariance* of the assets. The interpretation is that given a sufficiently high n, all asset specific risk can be diversified away. However, the non-specific or systematic risk, which affects all assets, will remain. This is referred to as *market risk*.

The Efficient Frontier

ISSN: 2455-8834

Volume:06, Issue:10 "October 2021"

We can plot all individual assets and combination of assets in $(\bar{R}_a, \sigma^2)$ space, represented by the points either on or within the boundary in ***Fig.3***. We call this the *feasible set* of portfolios. Only portfolios on the upward sloping part of the boundary (the part in bold connecting points MVP through A, B and C) are efficient. We call this part the *efficient frontier* (EF). The EF dominates all portfolios below it, both those off the boundary and those on its downward sloping part, since it offers either a lower risk for a given level of expected return or a higher expected return for a given level of risk. To illustrate, the point A' has the same level of risk as point A but offers a lower expected return and can therefore be excluded out hand as inefficient.

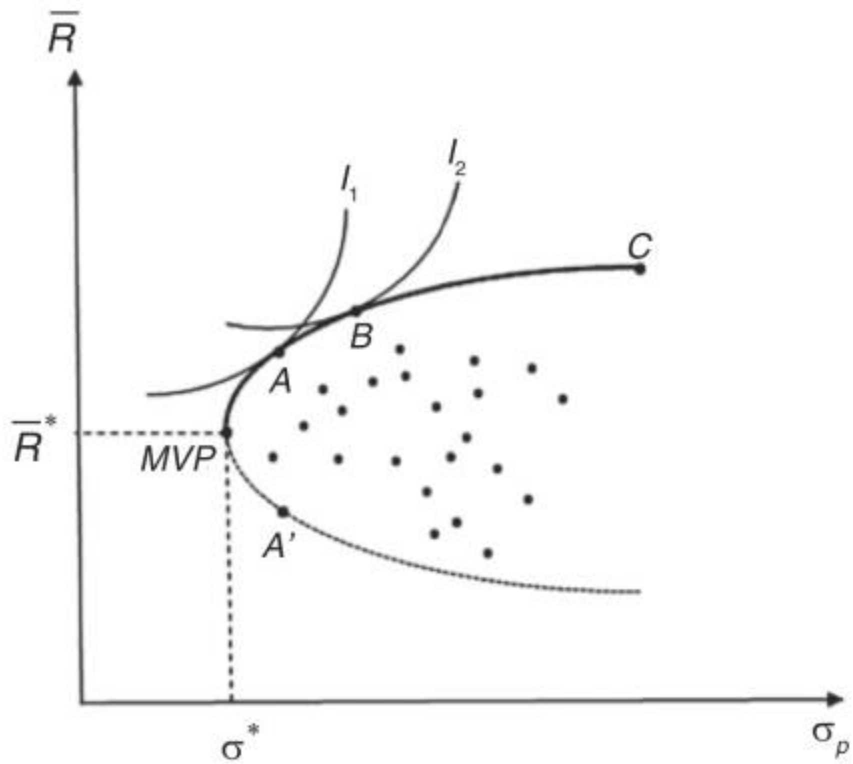

Fig.3 The efficient frontier

Which point on EF the investor choose will depends on their attitude to risk. This is illustrated in the figure by the two different indifference curves, I_1 and I_2. The points of tangency between the indifference curves and the EF (A and B) represent the optimum portfolios. But investors represented by I_2 have a greater tolerance for risk than those represented by I_1 and will therefore have more "aggressive" portfolios withboth a higher expected return and a higher volatility.

ISSN: 2455-8834

Volume:06, Issue:10 "October 2021"

The point MVP represents the portfolio with the minimum variance with a return of$\bar{R}^*$, we can trace our the efficient frontier. Our optimization problem will therefore must be equal to some attainable target,$\bar{R}_p$. the second constraint is that the portfolio must be fully invested:

$$\min_{\{\omega_1,\cdots,\omega_1\}} \sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i\, w_i \sigma_{ij}$$

subject to:

$$\sum_{i=1}^{n} w_i \bar{R}_i = \bar{R}_p$$

And

$$\sum_{i=1}^{n} w_i = 1$$

Forming the Lagrangian, we get:

$$L = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i\, w_i \sigma_{ij} + \lambda_2\left(\sum_{i=1}^{n} w_i \bar{R}_i - \bar{R}_p\right) + \lambda_1\left(\sum_{i=1}^{n} w_i - 1\right) \qquad \text{(A.25)}$$

Our equation has n+2 unknows. Taking the partial derivatives of L with respect to w_{ii}, and λ_1 and λ_2 and setting the first derivatives equal to zero, as we can solve forthe n+2 unknowns.

A Portfolio of Three or More Risky Assets

The Mean-Variances optimization model with two assets could be extended to more assets easily since the basic idea and function are similar.

Ledoit Wolf Shrinkage

Nowadays, it is universally recognized that the sample covariance matrix contains estimation error of the most likely to perturb a mean-variance optimizer, which is a phenomenon called “error-maximization” by Michaud (1989). So, it’s obsolete to use it for the purpose of portfolio optimization. Thus, the Ledoit Wolf method is used to obtain the sample covariance matrix

ISSN: 2455-8834

Volume:06, Issue:10 "October 2021"

through a transformation called *shrinkage*. This tends to pull the most extreme coefficients towards more central values, thereby systematically reducing estimation error where it influences most. The most important and challengeable part is to obtain the optimal shrinkage intensity, whose formula willbe given below. Without changing any other step in the portfolio optimization process, it's demonstrated on actual stock market data that shrinkage reduces tracking error relative to a benchmark index, and substantially increase the realized information ratio of the active portfolio manager.

Shrinkage Principle

Unlike other approaches such as using a combination of industry factors and risk indices or using statistical factors like principal components, Ledoit proposes to find a compromise between the sample covariance matrix S and a highly structured estimator F by computing a convex linear combination $\delta\delta F + (1 - \delta\delta)S$, where $\delta\delta$ is a number between 0 and 1 referred to as the *shrinkage constant*.

Shrinkage Target

The shrinkage target should fulfill the two requirements simultaneously: involving only a small number of free parameter (which is lots of structure) but also reflecting significant characteristics of the unknown quantity being estimated. Ledoit and Wolf has suggested to use the single-factor matrix Sharpe as the shrinkage target. However, in this paper, we propose another one: the *constant correlation model*.

Below is the derivation of the shrinkage target F:

Firstly, let y_{it}, $1 \leq i \leq N, 1 \leq t \leq T$, denote the return on stock i during period t. I assume that the stock return is independent and identically distributed (iid) over time and have finite fourth moments.

The sample average of the returns of stock i is given by $\bar{y}_t = T^{-1} \sum_{t=1}^{T} y_{it}$.

The population covariance matrix is denoted by Σ and the sample covariance matrix isdenoted by S. Typically, entries of the matrix Σ and S are denoted respectively by σ_{iijj} and s_{iijj}.

The average population and sample correlations between the returns on stocks I and j are given by

$$\varrho_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}} \quad and \quad r_{ij} = \frac{s_{ij}}{\sqrt{s_{ii}s_{jj}}}$$

ISSN: 2455-8834

Volume:06, Issue:10 "October 2021"

The average population and sample correlations are given by

$$\bar{\varrho} = \frac{2}{(N-1)N}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\varrho_{ij} \quad and \quad \bar{r} = \frac{2}{(N-1)N}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} r_{ij}$$

Then define the population constant correlation matrix Φ by means of the populationvariances and the average population correlation:

$$\phi_{ii} = \varrho_{ii} \quad and \quad \phi_{ij} = \bar{\varrho}\sqrt{\varrho_{ii}\varrho_{jj}}$$

Correspondently, define the sample constant correlation matrix F bhy means of the sample varainces and the average sample correlation:

$$f_{ii} = s_{ii}$$

$$f_{ij} = \bar{r}\sqrt{s_{ii}s_{jj}}$$

Shrinkage Intensity

Then with some complex calculation we have the formula of estimated shrinkage target proven by Ledoit and Wolf:

$$\widehat{\delta^*} = max\left\{0, min\left\{\frac{\hat{k}}{T}, 1\right\}\right\}$$

4. Main result

Data rationality

Below is the Gaussian distribution of the return after algorithm. It's obvious to notice that the daily price fluctuations of all six stocks are relatively stable except from Cambricon (688258) which is a new company that has go IPO last year in the STAR[2] market (so it's more volatile).

ISSN: 2455-8834

Volume:06, Issue:10 "October 2021"

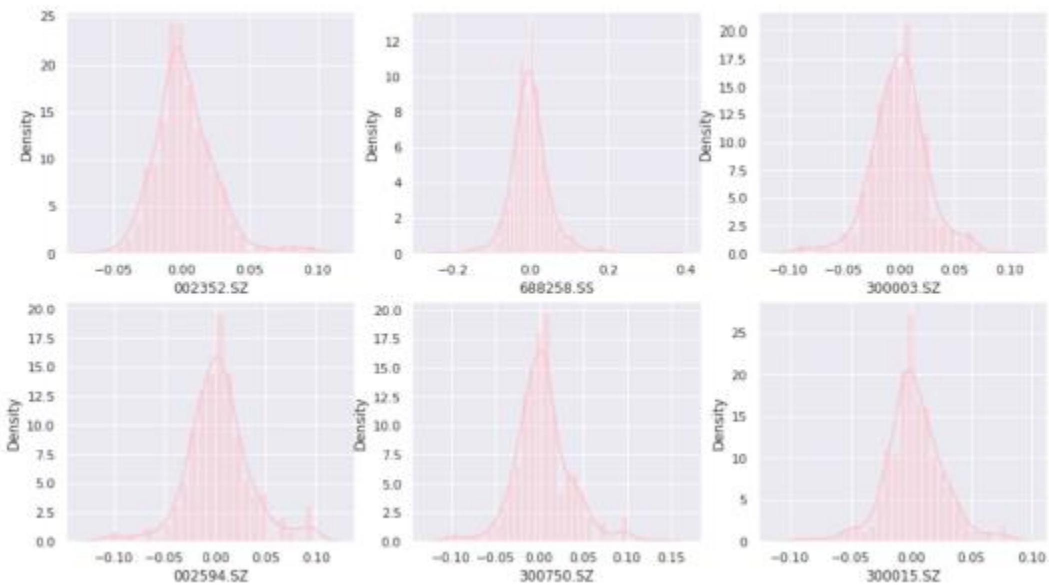

Fig.4 Gaussian Distribution of the return after algorithm

Below is the daily value of the return after algorithm in the span of a year (365). The trend is in tandem with the result above that the price of all five stocks except Cambricon are relatively stable.

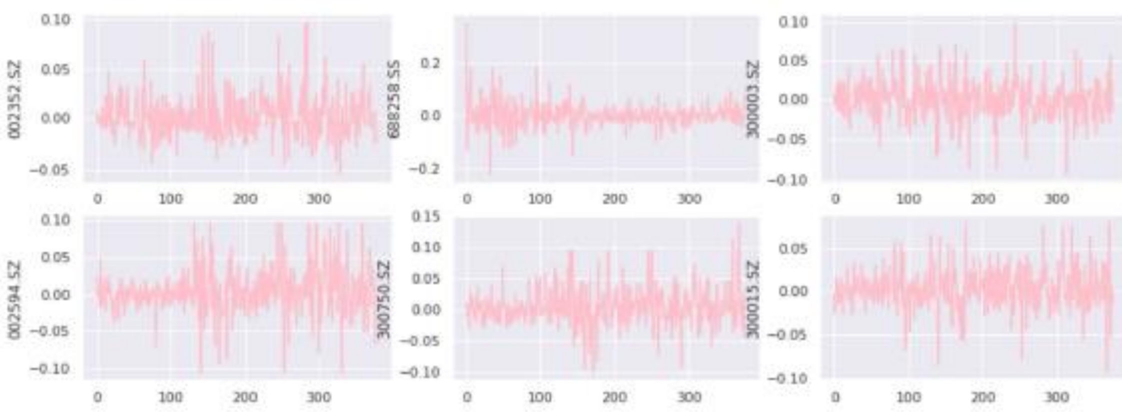

Fig.5 Fluctuation of return after algorithm

In general, based on the relatively stable performance of these 6 stocks we can deducethe market is in a normal and rational state that helps us to single out the random factor – market crisis/boom – out of our model.

ISSN: 2455-8834

Volume:06, Issue:10 "October 2021"

Efficient frontier

Below is the efficient frontier without Ledoit-Wolf shrinkage formed by 1000000 randomly generated portfolio represented by each green dot ($\bar{R}_p$, σ_p^2) .The red point noted as Opt is the optimization point after trade-off between risk parameter and expected return. All the 1000000 points are called the feasible set of portfolios. However, only portfolios on the upward sloping part of the boundary are considered as efficient since it offers either a higher expected return for a given level of risk or a lower risk for a given level of expected return. Thus, this part is what people called efficient frontier (EF).

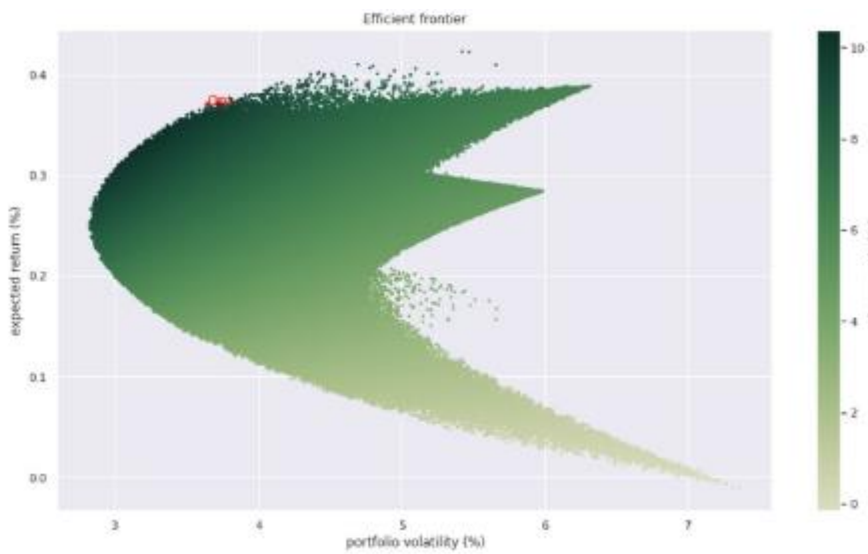

Fig.6 The efficient frontier without Ledoit-Wolf shrinkage

Optimal Weight	
Asset	Weight
CATL (300750)	1.791088e-01
BYD (002594.SZ)	3.025810e-01
Aier (300015)	4.659359e-17
S.F. (002352)	0.000000e+00
Cambricon (688258)	3.776072e-01
BOE (000725.SZ)	1.407030e-01

ISSN: 2455-8834

Volume:06, Issue:10 "October 2021"

Optimal Expected Return, Volatility & Sharpe Ratio	
Optimal Expected Return	0.004
Optimal Volatility	0.037
Optimal Sharpe Ratio	0.102

Efficient frontier with Ledoit-Wolf Shrinkage

Below is the efficient frontier with Ledoit-Wolf shrinkage formed by 1000000 randomly generated portfolio. The red point noted as Opt is the optimization point after trade-off between risk parameter and expected return

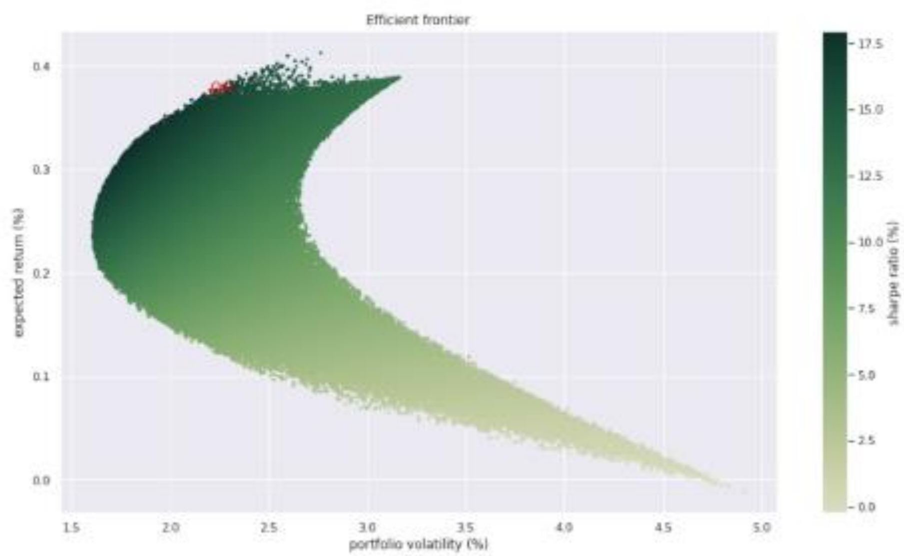

Fig.7 The efficient frontier with Ledoit-Wolf shrinkage

Optimal Weight	
Asset	Weight
CATL (300750)	1.662405e-01
BYD (002594.SZ)	3.130083e-01
Aier (300015)	9.584347e-17
S.F. (002352)	6.396793e-17
Cambricon (688258)	4.000000e-01
BOE (000725.SZ)	1.207512e-01

ISSN: 2455-8834
Volume:06, Issue:10 "October 2021"

Optimal Expected Return, Volatility & Sharpe Ratio	
Optimal Expected Return	0.004
Optimal Volatility	0.022
Optimal Sharpe Ratio	0.171

5. Discussion

Ledoit-Wolf Shrinkage is better

When comparing the ***Fig.6*** and ***Fig.7***, it's not hard to recognize that the *feasible set* with Ledoit-Wolf Shrinkage has a lower volatility in general since the covariance is more robust. The expected returns from 0.2 to 0.4 in ***Fig.7***, comparing to ***Fig.6***, tendto have a lower volatility.

Risk Aversion

I choose six different values of risk parameter (a = 0, 1, 5, 10, 50, 100) demonstrated in the below diagrams.

Firstly, it's easy to observe that there is a strong, positive correlation between risk aversion and expected return & volatility. The lower the risk parameter, the higher the expected return and volatility. However, the change of point Opt is obvious only when a is altering from 0 to 10. Based on the graph, it's easy to notice that when a = 0, the Opt locate near the point when expected return = 0.39. But when a increases to 5, the Opt move near to location when expected return = 0.35. Then reach the ***Global Minimum Variance*** when a is equal and greater than 10 near point when volatility is equal to 1.6.

ISSN: 2455-8834

Volume:06, Issue:10 "October 2021"

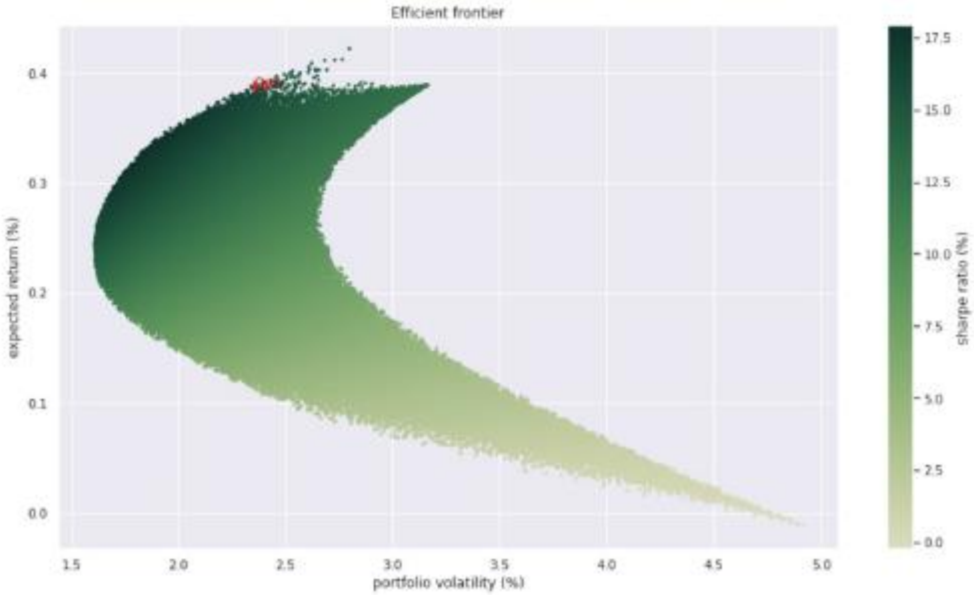

Fig.8 The efficient frontier when a = 0

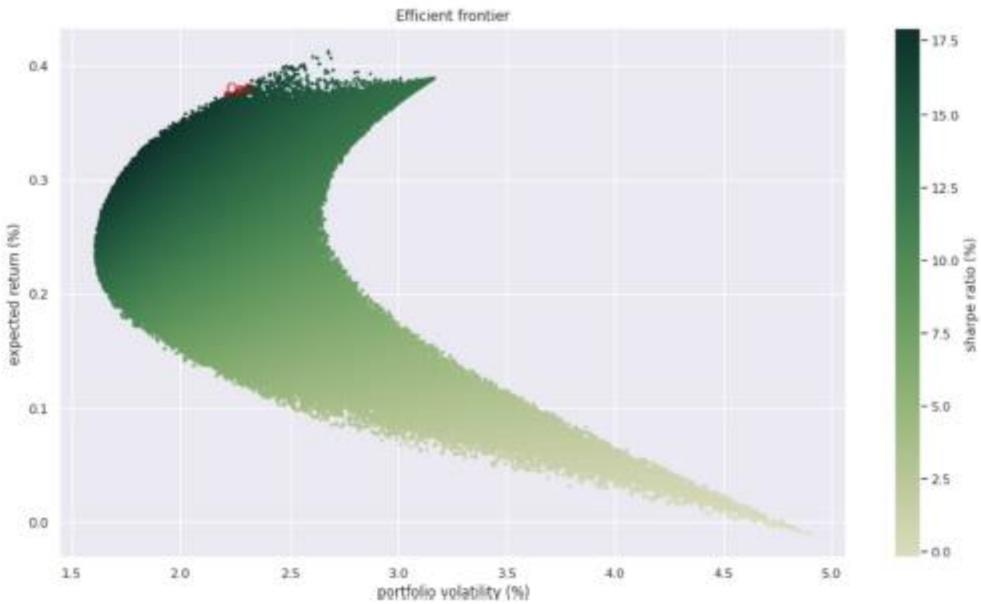

Fig.9 The efficient frontier when a = 1

ISSN: 2455-8834

Volume:06, Issue:10 "October 2021"

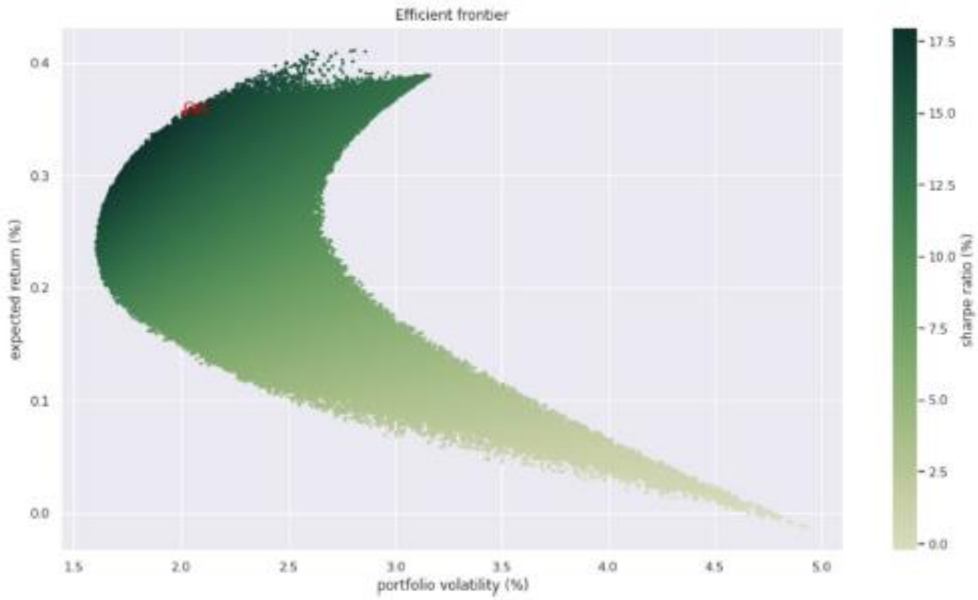

Fig.10 The efficient frontier when a = 5

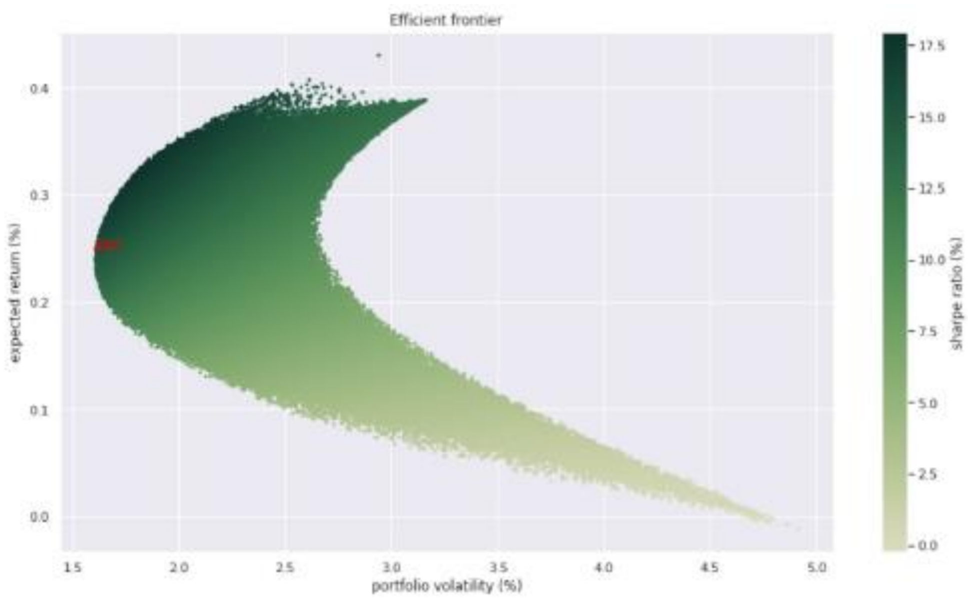

Fig.11 The efficient frontier when a = 10

ISSN: 2455-8834

Volume:06, Issue:10 "October 2021"

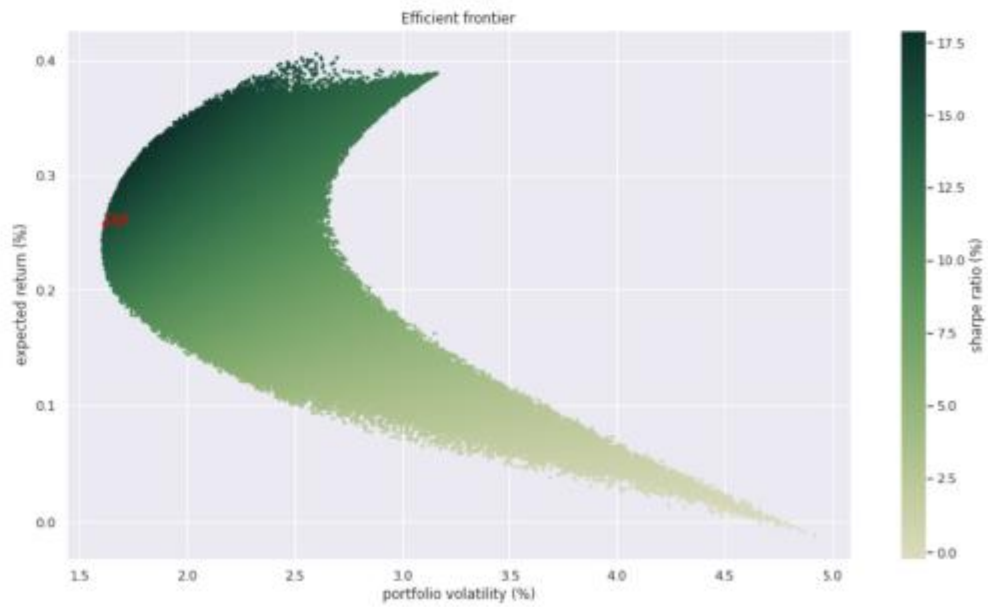

Fig.12 The efficient frontier when a = 50

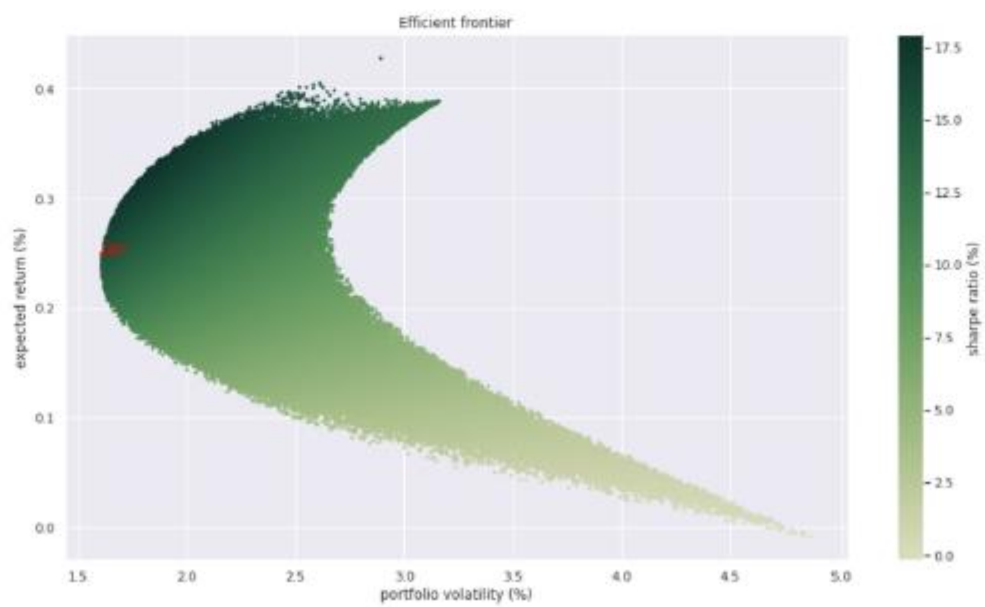

Fig.13 The efficient frontier when a = 100

Upper boundary for each stock

ISSN: 2455-8834

Volume:06, Issue:10 "October 2021"

I choose 5 different values of Upper boundary of each stock in the portfolio (0.3, 0.4, 0.6, 0.8) to observe how this factor changes the final optimal portfolio. It's obvious to see that there is a strong positive correlation between the upper boundary and the optimal expected return that the higher the upper boundary of each stock, the higher the optimal expected return. In the figures below, the Opt seems to move along the efficient frontier corresponding to the change of upper boundary with the only exception when the upper bound equals to 0.2. The reason why Opt is located within the efficient boundary when the upper bound equals to 0.2 is because the total numberof the stocks and one with high volatility. Since the weight of each six stocks has to add up to 1 and five of them are relatively stable, the weight of each five stocks have to be 0.2 with the one (688256) near zero. So, the stock which has more stability and profitability cannot occupy more share of the whole portfolio to reach the mostefficient point. Thus, in this condition the Opt is located within the efficient frontier.

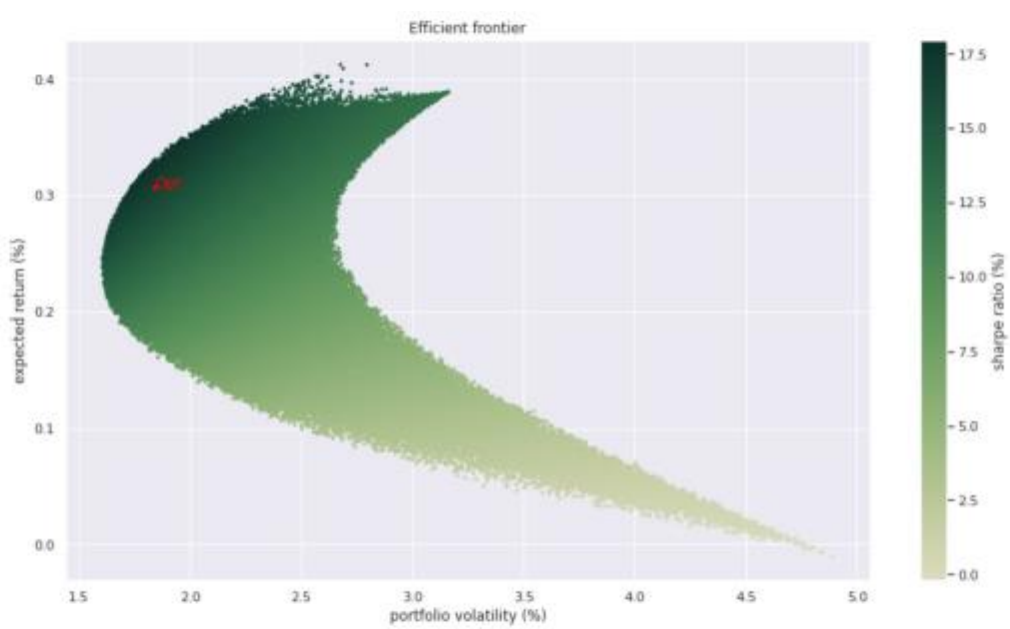

Fig.14 The efficient frontier when Upper boundary = 0.2

International Journal of Social Science and Economic Research

ISSN: 2455-8834

Volume:06, Issue:10 "October 2021"

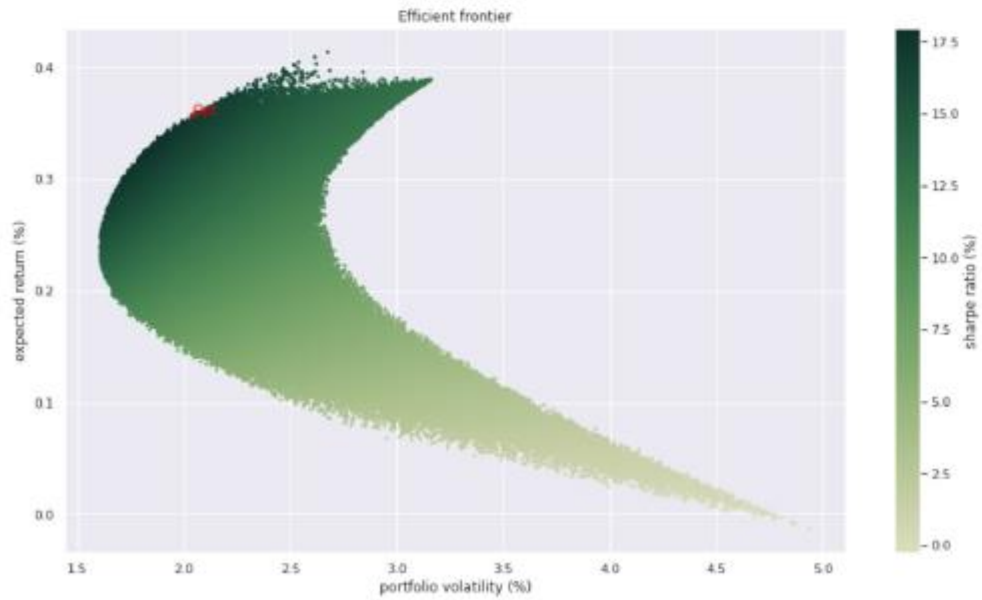

Fig.15 The efficient frontier when Upper boundary = 0.3

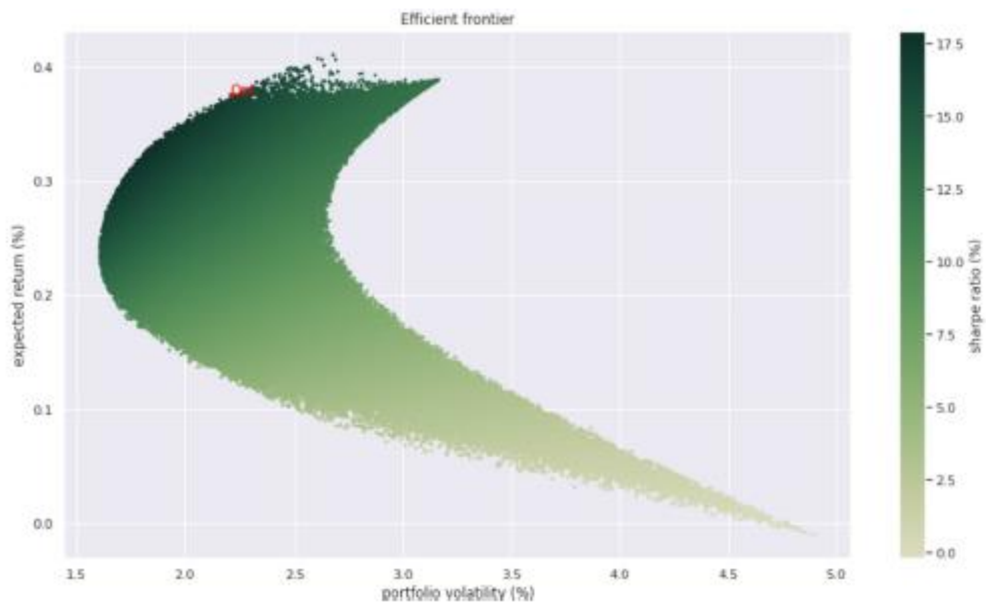

Fig.16 The efficient frontier when Upper boundary = 0.4

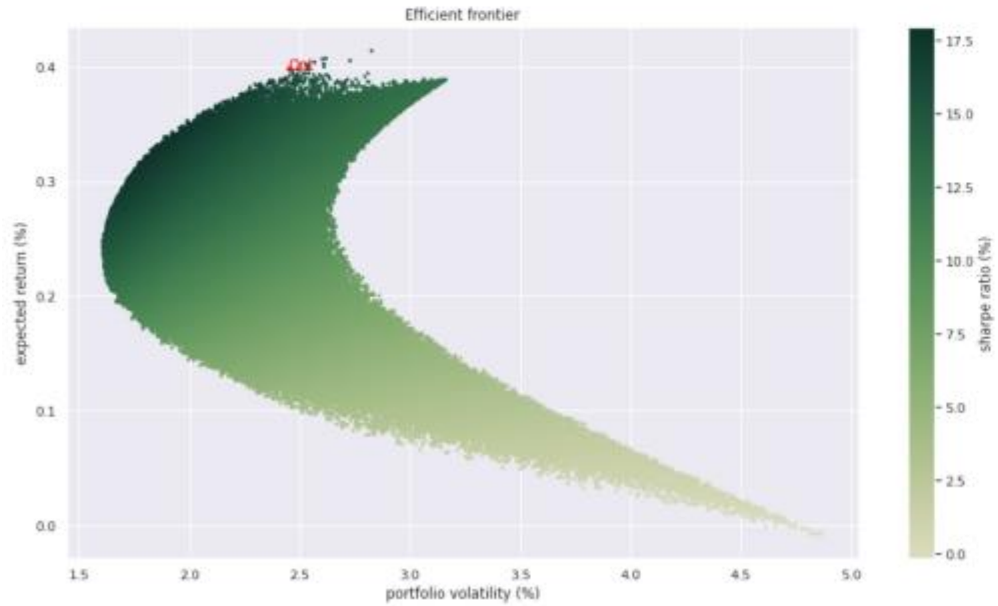

Fig.17 The efficient frontier when Upper boundary = 0.5

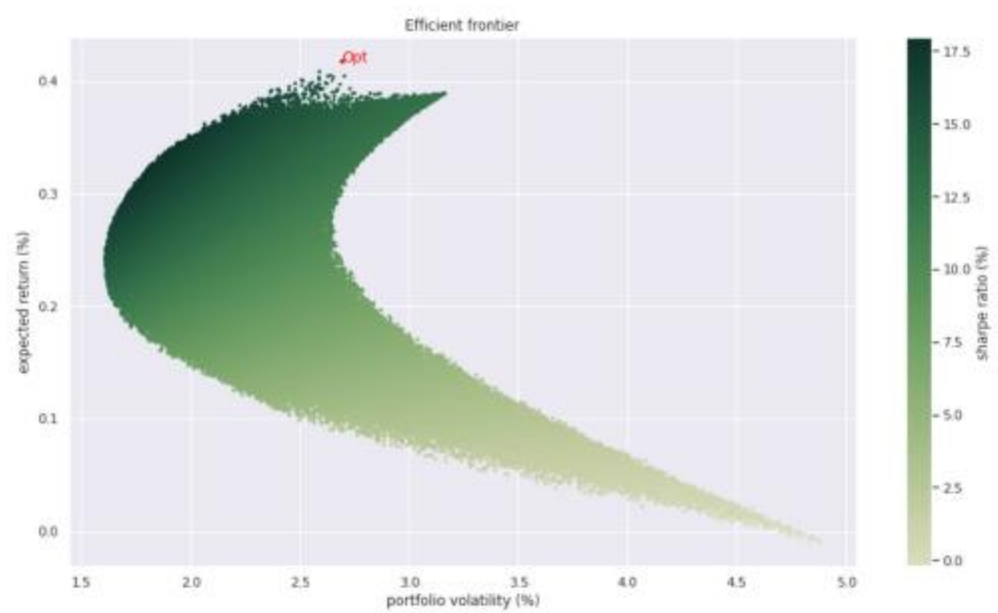

Fig.18 The efficient frontier when Upper boundary = 0.6

6. Conclusion

ISSN: 2455-8834

Volume:06, Issue:10 "October 2021"

In conclusion, the CAPM is an efficient tool to help the investor to optimize aportfolio given that it's impossible for human to calculate all the possible weights and find the optimal one. Finding the best one red point out of hundreds and thousands of green points in the efficient frontier is like finding a needle in the ocean.

Moreover, the adoption of the Ledoit-Wolf Shrinkage will largely increase the robustness of the covariance which will further optimize the portfolio by reducing the volatility given the same level of expected return.

The CAPM can also adjusted its optimal portfolio based on the preference of the clients in terms of their risk aversion and restriction of upper boundary for each stock. This personalized feature enables CAPM to become a useful tool with great flexibilityfor every investor during the process of portfolio construction.

Potential improvements

However, there are some potential improvements that may be taken into consideration in the future.

1. Instead of using absolute return directly, relative optimization may be taken into consideration, which calculate the return of each stock by subtracting the benchmark return from the actual return. Thus, the results can further help the investor to make investment decision with a benchmark to compare.

2. Ignorance of the tail risk, which is the financial risk of portfolio assets moving more than three standard deviations from its current price, above the risk of a normal distribution (often caused by low-probability events refers to as "Black swan" event like Corona viruses).

Reference

1. Elton, Edwin J. and Gruber, Martin J. and Spitzer, Jonathan F., Improved Estimates of Correlation Coefficients and Their Impact on the Optimum Portfolios (2004). NYU Working Paper No. S-DRP-04-02, SSRN: https://ssrn.com/abstract=1295853

2. Ledoit, Olivier and Wolf, Michael, Honey, I Shrunk the Sample Covariance Matrix(June 2003). UPF Economics and Business Working Paper No. 691, Available at SSRN: https://ssrn.com/abstract=433840 or http://dx.doi.org/10.2139/ssrn.433840

3. Mean-Variance Optimization and the CAPM

ISSN: 2455-8834

Volume:06, Issue:10 "October 2021"

http://www.columbia.edu/~mh2078/FoundationsFE/MeanVariance-CAPM.pdf

4. Roche, Cullen O., Understanding Modern Portfolio Construction (February 22, 2016).SSRN: https://ssrn.com/abstract=2740027 or http://dx.doi.org/10.2139/ssrn.2740027

