# BINOMIAL OPTION PRICING AND RISK-NEUTRAL PRICING

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## ABSTRACT

Initiated by the current popularity of buying stock and option, this paper talks about the topic of Binomial Option Pricing, which is an options evaluation method. Through the example of a European call option in IBM and a graphic demonstration of the option price, the basic background concept of option payoff (payoff and net payoff) and price bounds are introduced. Then, the Two-period Binomial Model is constructed through a binary tree and available market data. To increase the accuracy and minimize the potential deviation, we utilize Risk-Neutral Pricing through codes, which allows us to manipulate the number of steps in model. By controlling the number of steps, the models can not only provide data, but it can also illustrate a graphic demonstration, which gives a broad view of the convergence point. However, the realistic pricing still diverges from the theorical pricing that we obtain. By further research on the Binomial Pricing and Risk-Neutral Pricing, we find out the deviation may be cause by the general assumptions that are made to simplify the model and the fluctuation of data through time. However, a model can be constructed to observe and analysis the constant change of data, which may potentially increase the precision of the calculated pricing. The dynamic volatility and interest rate reflect the instability of the market, which may be considered both its charm and risk.

## Introduction

**Background & Problem Restatement:**

With the growing popularity of buying stock and investing, more people start putting their spare money into purchasing an asset through a stock. However, most people don't know how the stock work and what is an option, which resulted in poor risk management. I was really intrigued to know how the investment strategy, specifically how option work and how people determine

the pricing of an option, so I can potentially avoid some risks and make a profitable and successful deal. The binomial option pricing model is an options valuation method that uses an iterative procedure and allows for the node specification in a set period. In finance, option is a contract which conveys to its owner, the holder, the right, but not the obligation to buy or sell an underlying asset. The binomial option pricing model is related to the Black Scholes model and its mathematical formula.

It was invented in 1979 by John Cox (a well-respected finance professor), Mark Rubinstein (a financial economist), and Stephen Ross (also a finance professor) originally to be used as a device to illustrate and explain to students of Cox how the Black Scholes model works. Hsia (1983) proved that under the central limit theorem and some parameters, the binary tree model will converge to a continuous BS model. The binary tree method is simple and easy and has been widely extended so far. Hull and White (1988) modified the binary tree model by controlling variation and applied it to American option pricing and found that this method converges faster. Recently, Rubinstein (2000) compared the trigeminal tree model with binary tree model and found that the advantage of the former is that it has one more degree of freedom than the latter, which makes the stock price change and time segmentation independent to each other. I am exploring this project because of my interest in providing a better investment strategy and searching for accurate options pricing. Currently the project is an alternative to other options pricing models such as the Black Scholes model. In this project, I will utilize a database from barchart.com and Google Finance to find the correct option pricing through the Binomial Option Pricing Model.

**Methodology**

**General Definition:**

- Call - an option type that gives the owner the right to purchase an asset (the underlying asset) for a given price (exercise price) on or before a given date (expiration date)

- Put - an option type that gives the owner the right to sell an asset for a given price on or before the expiration date

- American exercise style - a type of exercise style that gives the owner the right toexercise the option on or before the expiration date.

- European exercise style – a type of exercise type that gives owner the right to exercise the option on or before the expiration date.

**Table of symbols:**

**Table 1: table of symbols and variables**

| | |
|---|---|
| $S\_T$ | Price of underlying asset at time T |
| K | Strike price |
| T | Time to maturity |
| r | Risk Free Rate |
| C | Call Price |
| σ | Volatility of underlying asset |
| B | Price of discount bond with face value \$1 and maturity T (clearly, $B \leq 1$) |
| a | The number of shares of the stock held |
| b | The dollar amount invested in the riskless bound |

Introduction of option:

Option payoff

The payoff of an option on the expiration date is determined by the price of the underlying asset and strike.

Example.: Consider a European call option on IBM with exercise price \$100. This gives the owner (buyer) of the option the right (not the obligation) to buy one share of IBM at \$100 on the expiration date. The payoff of an option is never negative since the action can be not exercise

**Table 2: the call option owner's payoff which depends on the share of pricing of IBM on theexpiration date.**

| IBM Price | Action | Payoff |
|---|---|---|
| 80 | Not Exercise | 0 |
| 90 | Not Exercise | 0 |
| 100 | Not Exercise | 0 |
| 110 | Exercise | 10 |
| 120 | Exercise | 20 |
| 130 | Exercise | 30 |
| : | Exercise | ST-100 |

Net payoff includes the cost of the option. Example: A European call on IBM shares with an exercise price of $100 and maturity of three months is trading at $5. The 3-month interest rate, not annualized, is 0.5%.

**Table 3: the call's net payoff for an example of a European call**

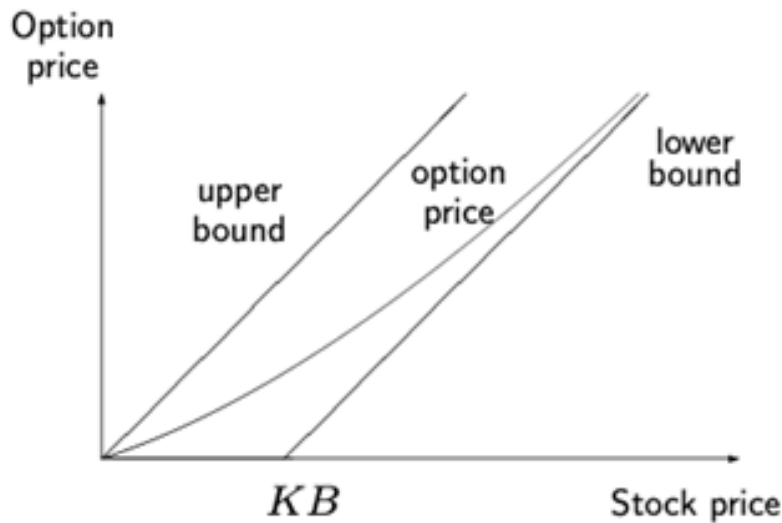| IBM Price | Action | Payoff | Net Payoff |
|---|---|---|---|
| 80 | Not Exercise | 0 | -5.025 |
| 90 | Not Exercise | 0 | -5.025 |
| 100 | Not Exercise | 0 | -5.025 |
| 110 | Exercise | 10 | 4.975 |
| 120 | Exercise | 20 | 14.975 |
| 130 | Exercise | 30 | 24.975 |
| : | Exercise | ST-100 | ST-100-5.25 |

**Price bounds**



**Figure 1: the option price curve in comparison to the upper bound and lower bound line First.**

we know that C>=0. Then, we can graph the payoff of stock dominates that of call. We utilize strategy (b) of buy a share of stock by borrowing KB. Since C>=0, we have C>=max[S-KB,0]. Finally, we combine the above, we have max[S-KB,0] <=C<=S, as illustrated above.
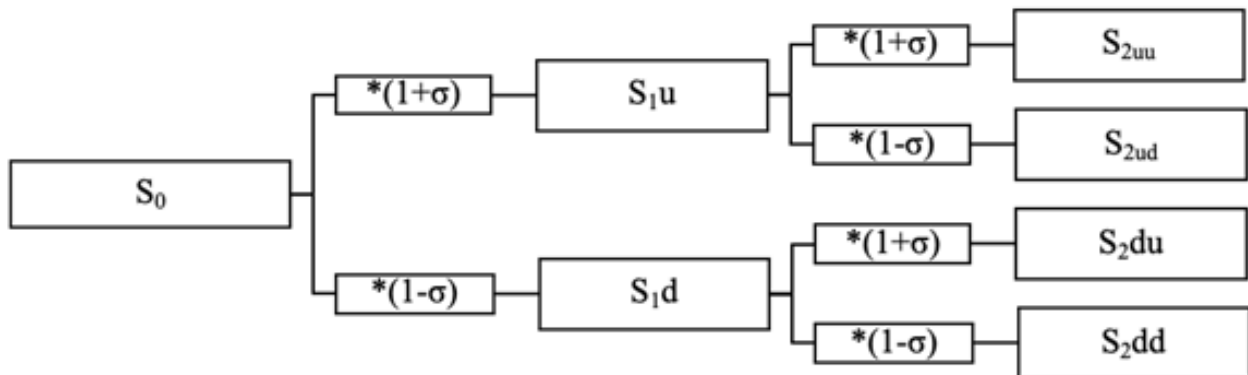
**Two-period Binomial Model:**



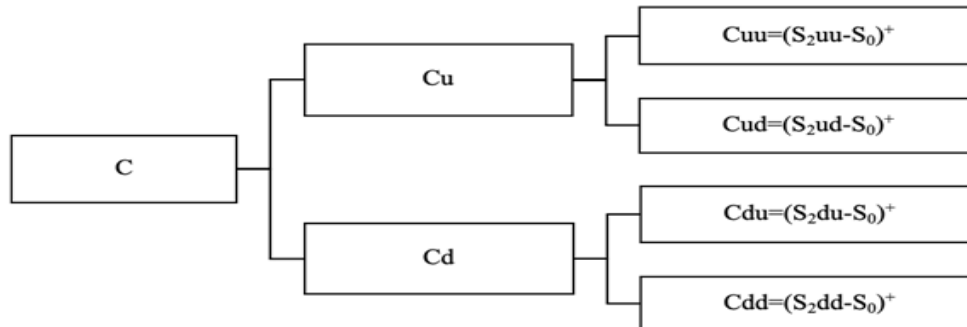**Figure 2: the stock price process of a two-period binomial model**

**Figure 3: the call price of a two-period binomial model ($x^+$=max (x.0))**

Cu and Cd denote the option value next period when the stock price goes up and goes down, respectively

Step 1: Starting with Period 1

1. Suppose the stock price goes to $S_1u$ in period 1

- Construct the replicating portfolio at node (t=1,up):

$$S_{2uu} \, a + (r+1) \, b = Cuu$$

$$S_{2ud} \, a + (r+1) \, b = Cud$$

- Then, we will get the unique solution of a and b

- The cost of this portfolio is (a) ($S_1u$) + b

- The exercise value of the option is the greater value btw the cost of portfolio and the difference with $S_0$ and $S_1u$

- Thus, we can find the value of Cu

2. Suppose the stock price goes to Cd in period 1

Repeat the above for node (t=1, down):

- The replicating portfolio (a and b)

- We can then calculate the lower node next period is the value of Cd

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Step 2 Now go back one period to Period 0:

- The option's value next period is either Cu or Cd depending upon whether the stock price goes up or down:

- If we can construct a portfolio of the stock and bond t replicate the value of the option next period, then the cost of this "replicating portfolio" must equal the option's present value

- Find a and b so that

$$(S_1u)\ (a) + (r+1)\ b = Cu$$

$$(S_1d)\ (a) + (r+1)\ b = Cd$$

- Then we can find the unique solution of a and b

- The cost of this portfolio is (a) $(S_0)$ + b = C0

- The present value of the option must be C0 (which is greater than the exercise value 0)

Implementation

"Risk Neutral" Pricing

Through the information in barchart.com and Google Finance, I extract the information of S (stock price), K (strike price of the option), T (time until expiry of the option), r (interest rate), vol (the volatility of the stock), N (number of steps in the model).
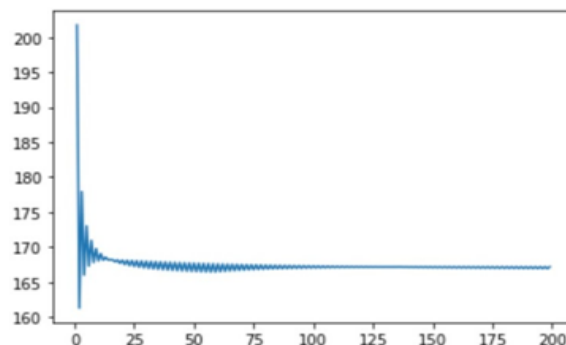


**Figure 4: the graph of option pricing from 1 to 200 steps in the model**

By inserting the number of steps in the model to 200, the convergence point is revealed at 100. At 100 on the x-axis, the fluctuation can be barely observed.
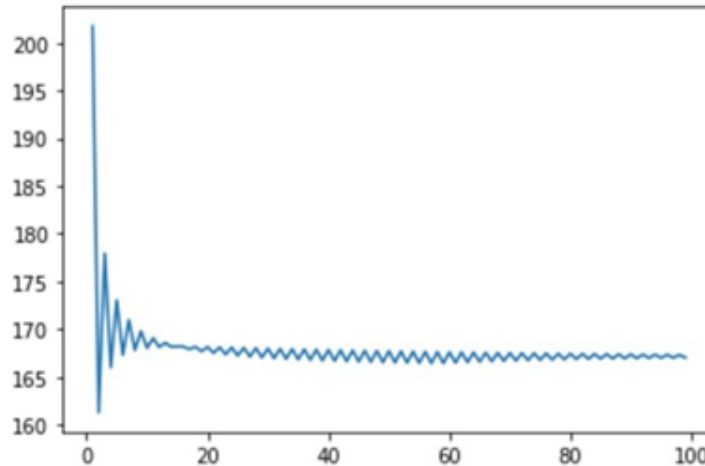


**Figure 5: the graph of option pricing from 1 to 100 steps in the model**

For a closer view, the data is minimized to 1 to 100 steps. It reveals the theoretical pricing, which approximately is 167.

**Result**

| Strike | Moneyness | Bid | Midpoint | Ask | Last | Change | %Chg | Volume | Open Int | Vol/OI | IV | Last Trade | Links |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4,650.00 | +0.99% | 149.20 | 149.85 | 150.50 | 144.31 | 15.42 | +11.96% | 1,482 | 165 | 8.98 | 17.54% | 15:47 ET | ⋮ |

**Table 4: the realistic pricing from barchart.com**

Our theoretical pricing is 167, the realistic pricing is around 149.20 to 250.50. The potential factors of deviation may come from the assumptions in the Binomial Option Pricing and Risk-Neutral Pricing. In the binomial option pricing, it includes the assumption that a price can only move to one of the two possibilities in each tree note, which isn't realistic. An asset can fluctuate to any price within the given range. In the Risk-Neutral Pricing, it cooperates the assumption of today's faire price of a derivative is equal to the expected value of its future payoff discounted by the risk-free rate. However, today's faire price of a derivative isn't necessary congruent to the future value. Dynamic interest rate and the volatility change may also contribute the difference between two pricings. These dynamic data bring the insight of constructing individual model to evaluate the change of volatility before the inserting risk-free model. Through the analysis of the

data itself, we can find out pattern in the volatility and substitute in the function, which could reduce the digression.

**Conclusion**

Binomial tree asset pricing can offer us a relatively accurate option price estimation with large steps. The theorical pricing found through the Risk-Neutral Pricing, but the realistic pricing from barchart.com reveals a different pricing. The possible explanations to the deviation can be the assumptions, limitation, and fluctuation in the models, bring the insight of inserting or cooperating a model to analysis the fluctuation of the data, such as volatility and interest rate.

This method can bring a potential more precise pricing relative to the realistic pricing, especially for long-term options. Through this research, the fluctuation in the market is revealed. The data has a great range of fluctuation through different ways, which reflect the potential risk in the market.

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