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DERIVATIVE PRICING USING BINOMIAL TREE SIMULATION

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ABSTRACT

Since financial derivatives have been invented for hedging, it has grown tremendously in assets amount. Understanding how option derivative is priced could help investors to make better investment decision. In this project, I study the binomial asset pricing model for stock prices. The pricing model is visualized and validated with Monte Carlo simulation. Further, I employed the binomial tree asset pricing model and Monte Carlo simulation to derive the price of JP.Morgan option price. The outputted model price matches almost exactly the option price we observe in the financial market. The studies thus might justify a potential arbitrage method to exploit the mid-priced financial derivatives.

Binomial Asset Pricing

1. Motivation

Since financial derivatives have been invented for hedging investment risks, it has grown tremendously in assets amount. Among all the complicated financial derivatives, the option is one of the most popular products because of its nature hedge for downside risk and its simple payoff. Understanding how option derivative is priced could help investors to make better investment decision. For example, if the option price is higher than its theoretical value, an investor might make a profit by constructing a short position in his/her portfolio. Building up a model for option pricing from this perspective can thus help us to remain advantageous in the financial market. After we have the model set up, it will be put in through a series of lines of code into spyder and it will run through a simulation that will estimate the price of the stock market. In this project, I explore the binomial asset pricing framework and employed it for JP.Morgan called option pricing. This will involve a theory called the Random Walk theory and will be used to help us to access future stock prices along with the model, after which computer

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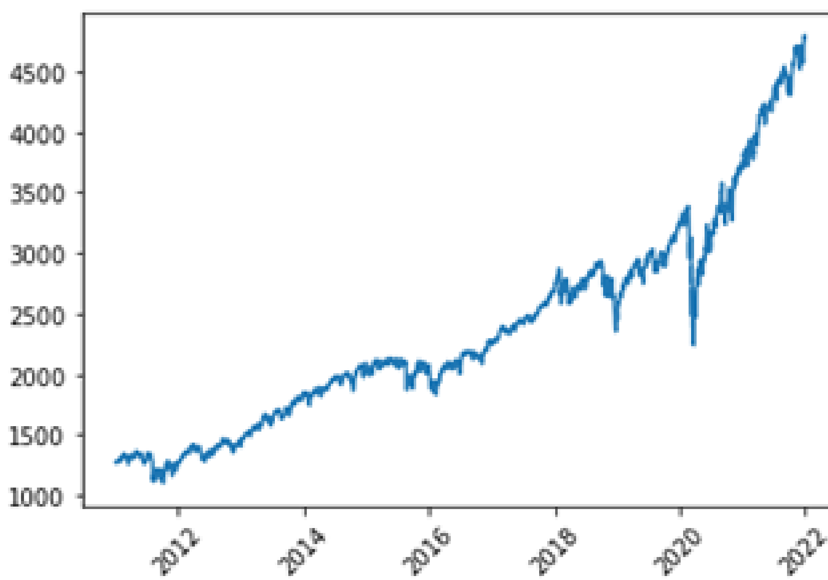
simulation can be run to help us determine the option prices. Studying and implementing such a model will help us understand the relationships between the risks and return of the assets in stocks.

2. Financial Data Visualization

To model the price of financial derivatives, one often starts by modeling the stock returns as the price of the financial derivatives highly depends on its underlying stock price, which can be derived from stock returns given the current price.

To further study the characteristic of financial returns, I plot the SP500 movement over the year 2021. I found that 1. the SP500 seems to be growing in the long turn with some averaged return u and that 2. There seems to be an averaged down-side return d for the day-to-day SP500 price movement.



Within a short time of period, one starting model might be to assume a random walk behavior for the stock price movement. The Random Walk is a theory that suggests that the difference in stock prices have an identical price distribution and is self-reliant on each other. Specifically, we can assume that in a short period of time, the stock price can either go up with percentage u or go down with percentage d with equal probabilities.

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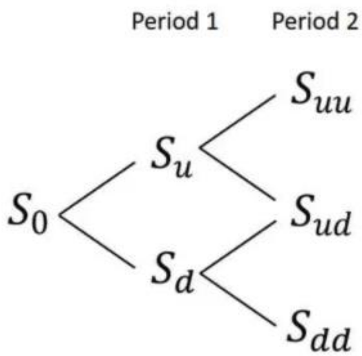

Where percentage u and percentage d are random with some expected value. The model then starts to behave like a binomial tree structure model. However, before I formally summarize the binomial tree model, I first studied some of the essential statistics concepts for the model.

3. Model

The Binomial Option Pricing Model is a risk-free method for calculating the value of path-dependent options. It is a popular tool for evaluating stock options, and investors use it to assess the right to buy or sell at specific prices over time. The model employs an iterative procedure that allows for the specification of nodes, or points in time, between the valuation date and the expiration date of the option. Before I delve into the details of the model, I first study some preliminary statistics that are relevant for estimating the model with historical stock data.

3.1 Some Definition

Mathematically, a random variable can take different values X_i with different probabilities P_i. It is useful to describe the random variable with its expectation and variance

- Expectation

$$E[X] = \sum_{i=1}^{n} \quad X_i P_i$$

where p_i is the probability for $X = X_i$. This is essentially the center or the average value for the random variable.

- Variance

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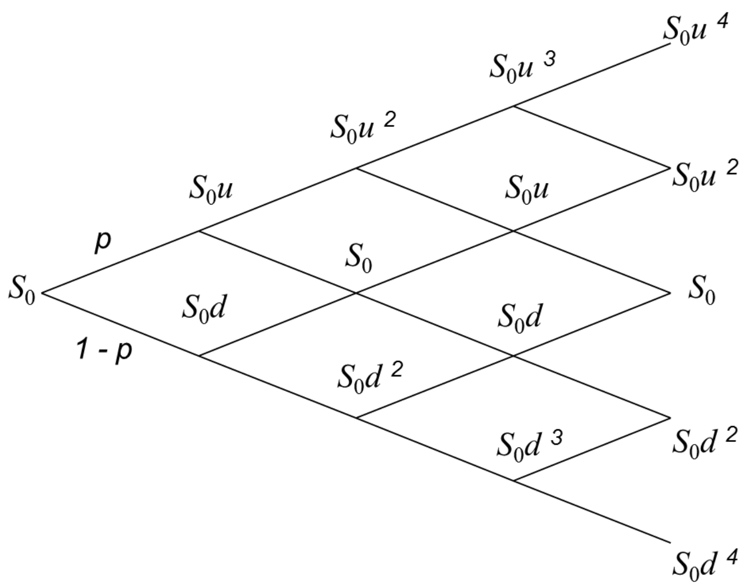
$$VaR[X] = \sum_{i=1}^{n} (X_i - E[X])^2 or\ E[X^2] - E^2[X]$$

The variance measures how concentrated is the random variable w.r.t it's center.

3.2 Model Setup

To translate the random walk behavior of stock price movement into a probability model,

we can denote the up percentage as u with probability p and down percentage as d with probability $1-p$, we could have the process is described with a coin tossing Bernoulli distribution and if we repeat the process many times, we could have the following tree sutures:



The problem is that once we know u, d and p, we could derive the possible outcomes that the future stock prices can take with different probabilities. Observe that for the future stock price, we just need to count how many times the stock price moves up and down. To better describe the probability of the possible outcomes of future stock prices, I study the binomial probability model.

3.3 Binomial Probability

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Suppose that after n time step, we reached some future time step T, then each step is equivalent to $\Delta = \frac{T}{n}$. Now if the stock price S_T is of our interest, we have could have S_T to be modeled by the following equation:

$$S_T = S_0 R_1 R_2 R_3 \dots R_n = S_0 u^x d^{n-x}$$

Where:

- x is the number of times the stock price moves up
- $n - x$ is the number of times the stock price moves down.

It is thus straightforward to observe that x follows a binomial distribution with parameter n and p:

$$X \sim Binomial(n, p)$$

According to the discrete random variable expectation and variance formula, we have

$$E[S_T] = E[S_0 R_1 R_2 R_3 \dots R_n] = E[S_0 u^x d^{n-x}] = \sum_{x=0}^{n} S_0 u^x d^{n-x} p(X = x)$$

$$VaR[S_T] = VaR[S_0 R_1 R_2 R_3 \dots R_n] = \sum_{x=0}^{n} S_0^2 u^{2x} d^{2n-2x} p(X = x) - E^2[S_T]$$

Specifically, for n =1, assume $p = \frac{1}{2}$ for simplicity, we have:

$$E[S_1] = \frac{1}{2} S_0 (u + d) VaR[S_1] = \frac{1}{2} S_0^2 (u^2 + d^2) - E^2[S_1]$$

From the reality, we could estimate the one-time step stock return with

$$E[R_1] = 1 + b\delta$$

$$VaR[R_1] = \sigma^2 \Delta$$

If we match up with the formula above, we could solve:

$$\frac{1}{2} S_0 (u + d) = S_0 (1 + b\Delta)$$

$$\frac{1}{2} S_0^2 (u^2 + d^2) - S_0^2 (1 + b\Delta)^2 = \sigma^2 \Delta$$

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And obtain:

$$u = 1 + b\Delta + \sigma\sqrt{\Delta}$$

$$d = 1 + b\Delta - \sigma\sqrt{\Delta}$$

Since we only need to match the mean and the variance asset pricing model to the data, we could alternatively model:

$$S_T = S_0 R_1^{\Delta} R_2^{\Delta} \dots R_n^{\Delta}$$

With

$$R_t^{\Delta} = (1 + b\Delta)(1 + X_t)$$

And

$$X_t = \{\frac{\sigma\sqrt{\Delta}}{1 + b\Delta}\ with\ p = 1/2\ -\frac{\sigma\sqrt{\Delta}}{1 + b\Delta}\ with\ p = 1/2$$

3.4 Monte Carlo Simulation

The binomial framework provides a pricing formula for the underly stock price, however, we are particularly interested in the prices of the financial derivative, which can be considered as a function of future stock prices$f(S_T)$.E.g., the call option is defined with the following payoff:

$$C_T = max(S_T - K, 0)$$

Where K is the strike price determined when an investor purchase the call option. To compute the expected value of the payoff, one can plug in the expectation formula:

$$E[C_T] = E[(S_T - K, 0)\ ] = \sum_{i=1}^{m} \quad max(a - K, 0)P(S_T = a)$$

The formula is computable but when the time step becomes finer and finer, one might end up with too many possible cases indicated by the number m. To solve it numerically, the Monte Carlo simulation is often used to replace the theoretical result.

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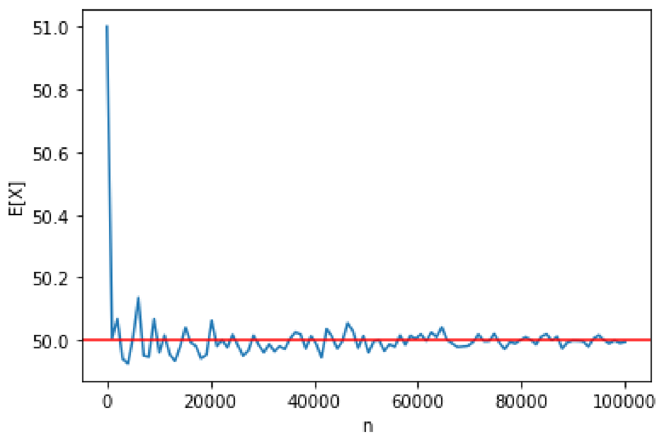
The Monte Carlo method is based upon the law of large numbers, that is, suppose $x_1, x_2, \dots x_n$ are drew indecently from the random variable X, it is known that as the sample size n approach infinity, the sample mean is close to its expectation.

$$\frac{1}{n}\sum_{i=1}^{n} x_i \approx E[X], as\ n \to \infty$$

For example, suppose we have $X \sim binomial(100,0.5)$ whose expectation equal to 100*0.5 = 50. If we sample observations of $x_1, \dots x_n$ and take the sample average $\underline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, the number $\underline{x}$should be close to 50 as we increase the sample size.



This provides us with a method to estimate the financial derivatives price through the following formula:

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) \approx E[f(X)],$$

In the call option price example, $x_i = S_T = S_0 R_1^{\Delta} R_2^{\Delta} \dots R_n^{\Delta}$, which can be simulated via binomial distribution, $f(x_i) = max(x_i - K, 0)$ and we can obtain:

$$\frac{1}{n}\sum_{i=1}^{n} max(x_i - K, 0) \approx E[C_T],$$

4. Binomial Return Simulation and Options Pricing

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4.1 Binomial Model Simulation

To validate the pricing framework, I firstly conduct binomial asset pricing for SP500 data, I obtained the stock price data for SP500 through Yahoo Finance. With a times series return available for $T = 189$ successive days between Jan/1st/2021 and Oct/1st/2021, I estimate the return expectation parameter b and σ^2 according to the textbook [1]:

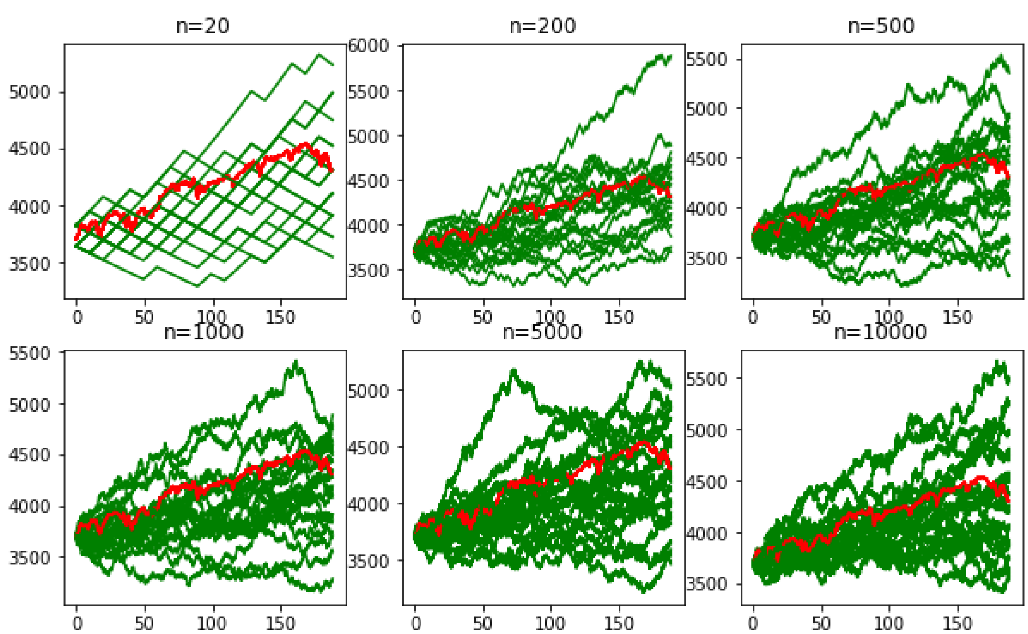
$$\hat{b} = Mean\big(log(S_t) - log(S_{t-1})\big)$$

$$\widehat{\sigma^2} = VAR\big(log(S_t) - log(S_{t-1})\big)$$

Then, with $\Delta = \frac{T}{n}$, I simulate the stock path through $S_T = S_0 R_1^\Delta R_2^\Delta \dots R_n^\Delta$ and $R_t^\Delta = (1 + b\Delta)(1 + X_t)$ with

$$X_t = \{\frac{\sigma\sqrt{\Delta}}{1 + b\Delta} \; with \; p = 1/2 \;\; -\frac{\sigma\sqrt{\Delta}}{1 + b\Delta} \; with \; p = 1/2$$

If we refine the time grids by increasing n, we can approximate the SP500 stock price movement with more and more realize the result. Below, I simulate 20 different paths across different choices of n, the red plotted line is the true SP 500 movement, which is the center of the 20 simulated paths:



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As we can see from the plot that as the time grids Δ go to 0 (or n goes to infinity), the stock movement gets more and more realistic. This is because, within a small step Δ, the stock movement finally approximately equal to a normal distribution with infinitely many possible choices and probabilities, which is the so-called Black-Shole pricing model [3].

4.2 Option Pricing

From the end of the above simulation, we could obtain a list of 20 simulated stock prices S_T, more importantly, the 20 simulated stock prices S_T are simulated according to the correct binomial probability assumption. The option price can then be computed as $\sum_{i=1}^{20} \quad \frac{1}{20} max(S_T^i - K, 0)$. If we increase the number of simulated paths from 20 to infinity, we should have the option price to be exact.

For implementation, I obtained the JPM option data from Yahoo Finance. Specifically, for today (Feb 22th 2022) the JP. Morgan option with ticker "JPM240119C00115000" has a value of \$43.0 with an implied volatility of 29.91%. I inputted \$115 strike K, 152.14\$ for the current price S_0, maturity T = 700/365, 1.47% for two years risk-free rate, and adopted simulation grids of N = 2,000 with repeated 10,000 simulations of the stock movement path to my binomial asset pricing model. The model outputted price is 41.34, which is almost exact to the true option price.

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